Metric polynomial: Let $f(x) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$ where $A_0, A_1, \ldots, A_n$ are the square matrix of the same order. Then $f(x)$ is said to be the metric polynomial.

Note: The degree of the metric polynomial is n.

For ex: $$f(x) = \begin{pmatrix} 1+x+x^2 & 2x \\ 3x+x^2 & 1+x^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+x+x^2 & 0\cdot1 + 2x + 0\cdot x^2 \\ 1\cdot0 + 3x + x^2 & 1 + 0\cdot x + 1\cdot x^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x^2 + \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is the metric polynomial of degree 2.

Simple polynomial: Let $f(x) = c_0 + c_1x + \cdots + c_nx^n$, where $c_0, c_1, \ldots, c_n$ are constants. Then $f(x)$ is said to be the simple polynomial.

For example: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now,

$$A - xI_3 = \begin{pmatrix} 1-x & 0 & 0 \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{pmatrix}$$

$$\therefore |A - xI_3| = (1-x)^3 = -x^3 + 3x^2 - 3x + 1.$$

Comparing the given equn with

$$c_0x^3 + c_1x^2 + c_2x + c_3$$

$$\therefore c_0 = -1 = (-1)^3$$

$$c_1 = 3 = 1+1+1 = \text{tr}(A)$$

$$c_3 = 1 = \det(A)$$

$$c_2 = -3 = (-1)^{3-2}\left[\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right]$$

$$= (-1)^{3-2}\left(\text{sum of the principle minors of order 2} \right.$$

Characteristic Equation:

Let A be an $n \times n$ matrix over a field F. and $\det(A - xI_n)$ is called the characteristic polynomial of A. Then $\det(A - xI_n) = 0$ is called the characteristic equation of A.

Then n-roots of the equation $\det(A - xI_n) = 0$ is called the characteristic roots of A.

Let, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$.

Now, $|A - xI_n| = \begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-x \end{vmatrix}$

$= c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n x^n$,

Where $c_0 = (-1)^n$,

$c_1 = (-1)^{n-1} \times (a_{11} + a_{22} + \cdots + a_{nn})$.

$c_n = (-1)^{n-n} \times$ (Sum of the principle minor of A of order A)

$c_n = \det(A)$.

Note: By the fundamental theorem of classical algebra, the ch. equation has atleast n roots (not necessarily distinct)

Usually denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$.

$\therefore |A - \lambda I_n| = c_0 x^n + c_1 x^{n-1} + \cdots + c_n$

$= (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) = 0$

Lemma: The constant term and the co-efficient of x^{n-1} in $\det(A - xI_n) = 0$ are $\det A$ and $\text{tr}(A)$.

Trace of a matrix: The sum of the characteristic roots of the matrix A is called the trace A and is denoted by $\text{tr}(A)$. (or The sum of the diagonal entries of the matrix A is equal to $\text{tr}(A)$).

Th: The product of the characteristic roots of the matrix A is $\det A$.

→ **Proof:** Let A be an $n \times n$ matrix and let the characteristic equation of A be $c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0$

Since the degree of the ch. equation is n, by fundamental theorem, the ch. eqn has n roots, say $\lambda_1, \lambda_2, \cdots, \lambda_n$.

We know that, the product of the ch. roots is equal to

$\lambda_1, \lambda_2, \cdots, \lambda_n = (-1)^n \frac{c_n}{c_0}$.

Here, $c_0 = (-1)^n$, therefore

$\lambda_1 \lambda_2 \cdots \lambda_n = c_n = |A|$.

Q) The ch. roots of the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ are

(a) 1, 1, 5 (b) 2, -1, 6 (c) 3, 4, 2 (d) 1, 1, 3.

Soln: We know that, the sum of the diagonal entries of the matrix A is $\text{tr}(A) = 1 \cdot 3 + 3 + 1 = 7$.

Now, $|A| = 5$.

We know that, the product of the ch. roots is equal to $\det A$. ∴ (a) (1, 1, 5) is true.

6) Let A be an $n \times n$ matrix and let the ch. equation of A be $c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0$. Then c_n is ——

(a) $-|A|$ (b) $-|A|^n$ (c) $|A|$ (d) 0

→

Th: 0 is a characteristic roots of the matrix A if A is singular.

→ Let A be an $n \times n$ matrix and let $|A - xI_n| = 0$ be the ch. equation of A i.e

$$c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0 \quad \text{---------- (1)}$$

Hence, $c_0 = (-1)^n$, $c_n = |A| = 0$ $\left(\because |A| = 0\right)$

i.e $c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x^{n-(n-1)} = 0$

$\Rightarrow c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x^n = 0$

$\Rightarrow$ 0 is a ch. roots of the eqn (1).

7) The characteristic roots of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$ are

(a) 3, 0, 0 (b) 1, 1, 1 (c) 3, −1, 2 (d) 4, −3, 2.

→ Since $|A| = 0 \Rightarrow$ 0 is a ch. roots of A.

Note: If A is a diagonal matrix, then the ch. eqn of A is equal to $(d_1 - x)(d_2 - x) \cdots (d_n - x)$, where $d_1, d_2, \cdots d_n$ are diagonal elements of the matrix A and the ch. roots of the matrix A are $d_1, d_2, \cdots d_n$.

Annihilates: Let A be an $n \times n$ matrix over the field F. A polynomial $f(x)$ is said to annihilates of the matrix A if $f(A) = 0$.

For ex: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $|A - xI_2| = (1-x)^2$

$= x^2 - 2x + 1$

Let $f(x) = x^2 - 2x + 1$.

$\therefore f(A) = A^2 - 2A + I$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= 0$

$\therefore f(x)$ is annihilates.

Cayley - Hamilton theorem

Th: For any square matrix A, the characteristic polynomial of A is annihilates of A.

proof: Let A be an $n \times n$ matrix and let $|A - xI_n| = c_0x^n + c_1x^{n-1} + \cdots + c_n$.

$(A - xI_n)$ is a matrix polynomial in x degree 1. and adj $(A - xI_n)$ is a matrix polynomial in x of degree $(n-1)$ Since each elements of the co-efficients is a polynomial in x of degree $(n-1)$ at most.

Let $\text{adj}(A - xI_n) = B_0x^{n-1} + B_1x^{n-2} + \cdots + B_{n-1}$.

Where $B_0, B_1, B_2, \ldots B_{n-1}$ are the square matrix of the same order.

Now, we have $(A - xI_n)\,\text{adj}(A - xI_n) = |A - xI_n| \cdot I_n$.

$(\because A \cdot \text{adj}A = |A|\, I_n)$

$\Rightarrow (A - xI_n)(B_0x^{n-1} + B_1x^{n-2} + B_2x^{n-3} + \cdots + B_{n-1}) = (c_0x^n + c_1x^{n-1} + \cdots + c_n)I_n$

Comparing the co-efficients of like powers of x, we get -

$-I_nB_0 = c_0I_n$

$AB_0 - B_1I_n = c_1I_n$

$AB_1 - B_2I_n = c_2I_n$

$AB_1 - B_2I_n = c_3I_n$

$\vdots$

$AB_{n-2} - B_{n-1}I_n = c_{n-1}I_n$

$AB_{n-1} = c_nI_n$

pre-multiplying the above equation, $A^n, A^{n-1}, A^{n-2}, \ldots I_n$ respectively and adding we get —

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_n I_n = 0$$ (proved).

Q) If $1, -2, 3$ be the characteristic roots of the matrix A, then find the ch. polynomial of A and hence find A^{-1}.

Soln: Since $1, -2, 3$ be the ch. roots of the matrix A.

∴ the ch. polynomial is

$$(x-1)(x+2)(x-3) = 0$$

$$\Rightarrow x^3 - 2x^2 - 5x + 6 = 0$$

∴ By cayley-Hamilton theorem, we get —

$$A^3 - 2A^2 - 5A + 6 = 0$$

Since $|A| \neq 0$, A^{-1} exist.

$$\therefore A^{-1}(A^3 - 2A^2 - 5A + 6) = A^{-1}\,0$$

$$\Rightarrow A^2 - 2A + 5I_3 + 6A^{-1} = 0$$

$$\Rightarrow 6A^{-1} = 2A + 5 - A^2$$

$$\Rightarrow A^{-1} = \frac{1}{6}(2A + 5 - A^2).$$

Q) If $1, -2, 3$ be the ch. roots of the matrix A, then A^{-1} is —

(a) $-\frac{1}{7}(A^2 - 2A - 5A^3)$ (b) $-\frac{1}{6}(A^2 - 2A - 5AI_3)$.

Q) Find the characteristic roots of the idempotent matrix A.

→ Since A is idempotent, $A^2 = A$.

∴ By cayley-Hamilton, theorem,

$$x^2 = x \text{ i.e. } x^2 - x = 0$$

$$\Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0, 1.$$

$\therefore$ 0 and 1 are the characteristic roots.

Th: Every idempotent matrix is singular without identity matrix.

(ii) If all the ch. roots, of the matrix A are zero, then A is nilpotent.

→ The ch. equation of the matrix A is

$$(x-0)(x-0)\cdots(x-0) = 0$$

$$\Rightarrow x \cdot x \cdots x = 0 \Rightarrow x^n = 0.$$

$\therefore$ By cayley - Hamilton theorem $A^n = 0$.

Similar matrices:

Let A be an $n \times n$ matrix over the field F. A is said to be similar to an $n \times n$ matrix B if $\exists$ a non-singular matrix P s.t

$$B = P^{-1}AP$$

Th: Similar matrix have the same characteristic polynomial.

→ **proof:** Let A and B be two similar matrix of order n. Then $\exists$ a non-singular matrix P s.t

$$B = P^{-1}AP.$$

The characteristic polynomial of A is $|A - xI_n|$.

We have to show that, $|A - xI_n| = |B - xI_n|$.

Now, $|B - xI_n| = |P^{-1}AP - P^{-1}(xI_n)P|$

$$= |P^{-1}(A - xI_n)P|$$
$$= |P^{-1}| \cdot |A - xI_n| \cdot |P|$$
$$= |A - xI_n| \cdot |P^{-1}| \cdot |P|$$
$$= |A - xI_n| \cdot |P^{-1} \cdot P|$$
$$= |A - xI_n| \cdot I_n$$
$$= |A - xI_n|.$$

Note: ① Similar matrix have the same ch. roots and also the det value of two matrix are same because the det value of the matrix is equal to the product of the ch. roots.

② The converse of the above theorem may not be true.

For example, Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

The ch. equation of A is $|A - xI_2| = 0$

$\Rightarrow (1-x)^2 = 0$

∴ the ch. roots of A are 1, 1 and the ch. equation of B is $|B - 4xI_2| = 0$

$\Rightarrow (1-x)^2 = 0$

Clearly, The ch. roots of B are also 1, 1.

Let P be a non-singular matrix.

Now, $P^{-1}AP = P^{-1}I_2P = P^{-1}P = I_2 \neq B$.

∴ B is not similar to A.

Q) If λ is a ch. roots of a non-singular matrix A, then show that the characteristic roots of A adjA is $\frac{|A|}{\lambda}$.

→ Since λ is a ch. roots of A, $|A - \lambda I_n| = 0$ ——— ①

We know that, $A \cdot \text{adj}A = |A| \cdot I_n$

$\Rightarrow I_n = \frac{A \cdot \text{adj}A}{|A|}$

From ①, $\left| A - \lambda \cdot \frac{A \cdot \text{adj}A}{|A|} \right| = 0$

$\Rightarrow \left| A \left(I_n - \frac{\lambda \, \text{adj}A}{|A|} \right) \right| = 0$

$\Rightarrow |A| \cdot \left| I_n - \frac{\lambda}{|A|} \cdot \text{adj}A \right| = 0$

$\Rightarrow |A| \cdot \left| \left(-\frac{\lambda}{|A|}\right) \left(\text{adj}A - I_n \cdot \frac{|A|}{\lambda} \right) \right| = 0$

$\Rightarrow |A| \cdot \left(-\frac{\lambda}{|A|}\right)^n \cdot \left| \text{adj}A - I_n \cdot \frac{|A|}{\lambda} \right| = 0,$

since $|eA| = e^{n}|A|$.

$\Rightarrow \left| \text{adj}\, A - I_n \frac{|A|}{\lambda} \right| = 0,$ since $|A| \neq 0$

∴ The ch. roots of adjA is $\frac{|A|}{\lambda}$.

For example: Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$.

Since A is an upper triangular matrix, the ch. roots of A are the diagonal elements i.e 2 and 5

Now, $|A|$ = product of the ch. roots
$= 2 \times 5 = 10$. and let $\lambda_1 = 2$, $\lambda_2 = 5$.

∴ The ch. roots of adjA are $\lambda_1' = \frac{|A|}{\lambda_1} = \frac{10}{2} = 5$

and $\lambda_2' = \frac{|A|}{\lambda_2} = \frac{10}{5} = 2$.

Q) If λ is a characteristic roots of a non-singular matrix A, then λ^{-1} is also a characteristic roots of A^{-1}.

→ Since λ is a ch. roots of the matrix A,

$$|A - \lambda I_n| = 0.$$

Now, $|A^{-1} - \lambda' I_n| = |A^{-1}(I_n - \lambda^{-1}A)|$

$= |A^{-1}(-\lambda^{-1})(A - \lambda I_n)|$

$= |A^{-1}| \cdot |(-\lambda^{-1})(A - \lambda I_n)|$

$= |A^{-1}| \cdot (-\lambda^{-1})^n \cdot |A - \lambda I_n|$

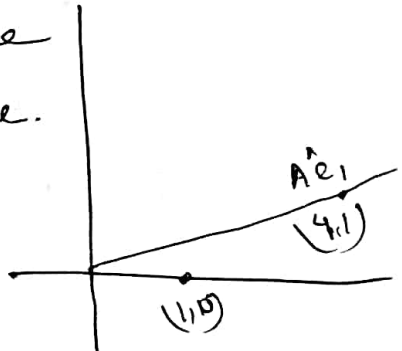
$= 0$

Geometrical meaning of Eigen values and Eigen vectors

Let $A = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$. and $\hat{e}_1 = 1\cdot\hat{i} + 0\cdot\hat{j} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Now, $A\hat{e}_1 = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

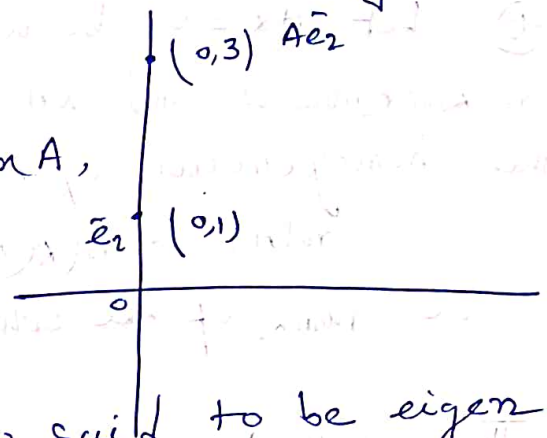
Here, $\hat{e}_1$ is not eigen vector, since $\hat{e}_1$ and $A\hat{e}_1$ is not the pt of same line.



Let $\hat{e}_2 = 0\cdot\hat{i} + 1\cdot\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$A\hat{e}_2 = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\hat{e}_2.$$

Hence $\hat{e}_2$ is an eigen vector corresponding to the eigen value $\lambda = 3$.



In general, if we have matrix A, can we find a vector x, s.t $Ax = \lambda x$.

☑ Defn: A number 'λ' is said to be eigen value of the matrix 'A' if $\exists$ a non-null vector x s.t $Ax = \lambda x$.

Any such non-null x is called an eigen vector of the matrix A corresponding to the eigen value λ.

Theorem: A number λ is a characteristic roots of the matrix A iff λ is an eigen value of A.

Note: (A) Two observation follows from the above theorem.

(i) λ is a ch. roots of the matrix A iff $|A - \lambda I_n| = 0$.

(ii) λ is an eigen value of A iff the homogeneous system $(A - \lambda I_n)x = 0$ has a non-trivial solution.

☑ (A) Let $Ax = 0$ be a homogeneous system of n-unknowns Then, (i) if $|A| \neq 0$, then the homogeneous system has a

unique solution. This is trivial solution $(0,0,\cdots,0)$.

(ii) if $|A| = 0$, then the homogeneous system has a non-trivial solution i.e infinite number of solutions.

(B) Let $Ax = 0$ be a homogeneous system of n-unknowns and xA be the solution space of the homogeneous system. Then,

$$r(A) + r(X(A)) = n \quad (\text{number of unknowns}).$$

i.e rank of the solution space $= n - r(A)$.

Th: Let A be an $n \times n$ matrix over the field F. For every eigen values corresponding to we get infinite number of eigen vector.

Soln: Let λ be an eigen value of the matrix A, $|A - \lambda I_n| = 0$.

Therefore, the homogeneous system of equation $(A - \lambda I_n)x = 0$ has a non-trivial solution.

Say $X = X_1$. Then, $(A - \lambda I_n)X_1 = 0$

$$\Rightarrow 1X_1 = \lambda X_1.$$

Since X_1 is arbitrary, therefore the eigen vector of A are infinite corresponding to the eigen value λ.

(6) Find the eigen values and eigen vectors of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

→ The ch. polynomial of A is

$$|A - xI_3| = 0$$

$$\Rightarrow (1-x)x(x-2) = 0$$

$$\therefore x = 0, 1, 2.$$

∴ Eigen values of the matrix A are 0, 1, 2 :

Let $X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector of the matrix A corresponding to the eigen value $\lambda = 0$.

Now, $(A - \lambda I_3) X_1 = 0$

$\Rightarrow AX_1 = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow \left.\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$

Let $x_3 = K$. Then $x_2 = -K$. and ~~Let x_1~~

$\therefore X_1 = \begin{pmatrix} 0 \\ -K \\ K \end{pmatrix} = K \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Let $X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector of the matrix A corresponding to the eigen value $\lambda = 1$.

Now, $(A - \lambda I_3) X_2 = 0$

$\Rightarrow (A - I_3) X_2 = 0$

$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow x_3 = 0, \quad x_2 = 0$

Let $x_1 = K$, K being arbitrary non-zero real number

$\therefore X_2 = \begin{pmatrix} K \\ 0 \\ 0 \end{pmatrix} = K \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Let $X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ for the eigen value $\lambda = 3$.

Now, $(A - 3I) X_3 = 0$

$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{aligned} x_1 &= 0 \\ -x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$

Let $x_2 = K$. $\therefore x_3 = K$

$\therefore X_3 = \begin{pmatrix} 0 \\ K \\ K \end{pmatrix} = K \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Definition: Let V be the vector space over the field F and W is non-empty subset of V. Then W is a subspace of V if

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha+\beta \in W, \forall \alpha, \beta \in W$.

(ii) $c \in F, \alpha \in W \Rightarrow c\alpha \in W \ \forall \alpha \in W, c \in F$.

Note: (i) $0 \in W$

(ii) The soln space of the homogeneous system is a subspace.

(iii) The soln space of the non-homogeneous system is not a subspace because $(0,0,0)$ does not belong to the soln space.

Characteristic subspace: Let A be an $n \times n$ matrix over the field F and λ be an eigen value of A.

The subspace of $V_n(F)$ consisting of all the eigen ~~value~~ vectors of A together with zero is called the characteristic subspace of the matrix A corresponding to the eigen value λ.

Its dimention is called the geometric multiplicity of the eigen value λ w.r.t A.

Note: $E_\lambda = \{ x \in V_n(F) : Ax = \lambda x \}$ is called the characteristic subspace of the matrix A corresponding to the eigen value λ.

We know that, $\rho(A - \lambda I_n)$ + rank of the ch. subspace of the matrix $(A - \lambda I_n) = n$ (number of unknown)

$\Rightarrow$ rank of the ch. subspace of the matrix $(A-\lambda I_n)$

i.e geometric multiplicity of $\lambda = n - \rho(A - \lambda I_n)$. $= n - \rho(A-\lambda I_n)$.

For example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. $\therefore A - \lambda I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Now, $\rho(A - \lambda I_3) = 2$ since $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$.

Now, g.m of $\lambda = 0$ is $[n - \rho(A - \lambda I_3)]$

$= 3 - 2 = 1$.

Let λ is an eigen value of A with ch. polynomial $f(x)$. The algebraic multiplicity of λ is the largest +ve integer K s.t $(x-\lambda)^K$ is a factor of $f(x)$.

Theorem: For any eigen value λ of the matrix A, the algebraic multiplicity of the ~~ma~~ eigen value λ w.r.t A is not less than the geometric multiplicity of λ w.r.t A.

→ The above theorem means, $1 \leq$ G.M $\leq$ A.M.

G.M → geometric multiplicity

A.M → arithmetric multiplicity.

Theorem: Two eigen vectors of a square matrix A over the field F corresponding to two different eigen value are linearly independent.

Proof: Let X_1 and X_2 be two eigen vector of the matrix A corresponding to two different eigen value λ_1 and λ_2. Then $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$.

Let us consider the relation,

$$c_1 X_1 + c_2 X_2 = 0, \text{ where } c_1, c_2 \in F \quad \text{——— (1)}$$

$$\Rightarrow c_1 AX_1 + c_2 AX_2 = 0$$

$$\Rightarrow c_1 \lambda_1 X_1 + c_2 \lambda_2 X_2 = 0 \quad \text{——— (2)}$$

From (1) and (2),

$$\begin{pmatrix} c_1 X_1 & c_2 X_2 \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} = 0 \quad \text{——— (3)}$$

Let $T = \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}$. Then $|T| = \lambda_2 - \lambda_1 \neq 0$, since $\lambda_1 \neq \lambda_2$.

$\therefore T^{-1}$ exist.

Post-multiplying (3) by T^{-1} we get —

$$\begin{pmatrix} c_1 X_1 & c_2 X_2 \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}^{-1} = 0$$

$$\Rightarrow \begin{pmatrix} c_1 X_1 & c_2 X_2 \end{pmatrix} I_2 = 0$$

$$\Rightarrow \begin{pmatrix} c_1 X_1, & c_2 X_2 \end{pmatrix} = (0, 0)$$

$$\Rightarrow c_1 X_1 = 0, \quad c_2 X_2 = 0 \quad \text{——— (4)}$$

Since X_1 and X_2 are non-trivial solution, therefore, from (4) $c_1 = 0$ and $c_2 = 0$.

$\therefore X_1$ and X_2 are linearly independent.

Theorem: The eigen values of a real symmetric matrix are all real.

→ **proof:** Let A be an $n \times n$ real symmetric matrix and let λ be an eigen values of the matrix A.

Then, $|A - \lambda I_n| = 0$, since λ is an eigen value of A.

Therefore, the homogeneous system of equation $(A - \lambda I_n)X = 0$ has a non-trivial solution. Say, $X = X_1$.

Then, $AX_1 = \lambda X_1$.

Taking transpose and conjugate, we get —

$$(\overline{AX_1})^T = (\overline{\lambda X_1})^T$$

$$\Rightarrow (\overline{X_1})^T (\overline{A})^T = (\overline{\lambda})(\overline{X_1})^T$$

$$\Rightarrow (\overline{X_1})^T A = (\overline{\lambda})(\overline{X_1})^T, \quad \text{since } (\overline{A})^T = A^T = A.$$

post multiplying by X_1, we get —

$$(\overline{X_1})^T A X_1 = (\overline{\lambda})(\overline{X_1})^T X_1$$

$$\Rightarrow (\overline{X_1})^T \lambda X_1 = (\overline{\lambda})(\overline{X_1})^T X_1$$

$$\Rightarrow \lambda (\overline{X_1})^T X_1 = \overline{\lambda} (\overline{X_1})^T X_1$$

$$\Rightarrow (\lambda - \overline{\lambda})(\overline{X_1})^T X_1 = 0$$

$$\Rightarrow \lambda - \overline{\lambda} = 0, \quad \text{since } X_1 \neq 0$$

$$\therefore \lambda = \overline{\lambda}.$$

Let $\lambda = a + ib$, $a, b \in \mathbb{R}$,

Then $\overline{\lambda} = a - ib$. Now $A\lambda = \overline{\lambda}$

$$\Rightarrow a + ib = a - ib$$

$$\Rightarrow b = 0$$

$\therefore \lambda = a \in \mathbb{R}$.

Therefore, the eigen values of a real symmetric matrix are all real.

Theorem:

The eigen values of a real skew symmetric matrix is either zero or purely imaginary.

Proof:

Let A be an $n \times n$ real skew-symmetric matrix and λ be an eigen value of A. Then $|A - \lambda I_n| = 0$, since λ is an eigen value of A.

Therefore, the homogeneous system of equation $(A - \lambda I_n)X = 0$ has a non-trivial solution.

Say $X = X_1$. Then, $AX_1 = \lambda X_1$.

Taking transpose and conjugate, we get —

$$\overline{(AX_1)}^T = \overline{(\lambda X_1)}^T$$

$$\Rightarrow (\bar{X_1})^T (\bar{A})^T = \bar{\lambda} (\bar{X_1})^T$$

$$\Rightarrow (\bar{X_1})^T (-A) = \bar{\lambda} (\bar{X_1})^T$$

$$\Rightarrow -(\bar{X_1})^T A = \bar{\lambda} (\bar{X_1})^T$$

Post multiplying by X_1, we get —

$$-(\bar{X_1})^T A X_1 = \bar{\lambda} (\bar{X_1})^T X_1$$

$$\Rightarrow -(\bar{X_1})^T \lambda X_1 = \bar{\lambda} (\bar{X_1})^T X_1$$

$$\Rightarrow -\lambda (\bar{X_1})^T X_1 = \bar{\lambda} (\bar{X_1})^T X_1$$

$$\Rightarrow (\lambda + \bar{\lambda}) (\bar{X_1})^T X_1 = 0$$

$$\Rightarrow \lambda + \bar{\lambda} = 0, \text{ since } X_1 \neq 0$$

$$\Rightarrow \bar{\lambda} = -\lambda.$$

Let $\lambda = a + ib$. Then $\bar{\lambda} = a - ib$. Now,

$$\bar{\lambda} = -\lambda$$

$$\Rightarrow a - ib = -(a + ib)$$

$$\Rightarrow a = 0$$

$\therefore \lambda = ib, \quad b \in \mathbb{R}$.

Therefore the eigen values of a real skew-symmetric matrix is either zero or purely imaginary.

Theorem: The eigen vectors of a real symmetric matrix corresponding to different eigen values is orthogonal.

→ **proof:** Let x_1 and x_2 be two eigen vectors of a real symmetric ~~no~~ $n \times n$ matrix A corresponding to two different eigen values λ_1 and λ_2.

Then $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$. ——— (2)
——— (1)

Taking transpose and conjugate of (1), we get —

$$(\bar{A}\bar{x}_1)^T = (\bar{\lambda}_1\bar{x}_1)^T$$

$$\Rightarrow (\bar{x}_1)^T(\bar{A})^T = \bar{\lambda}_1(\bar{x}_1)^T, \quad \text{since } A \text{ is real symmetric matrix}$$

$$\Rightarrow (\bar{x}_1)^T A = \lambda_1(\bar{x}_1)^T, \quad \bar{\lambda}_1 = \lambda_1$$

since $(\bar{A})^T = A^T = A$.

Now, post multiplying by x_2, we get —

$$(\bar{x}_1)^T A x_2 = \lambda_1(\bar{x}_1)^T x_2$$

$$\Rightarrow (\bar{x}_1)^T \lambda_2 x_2 = \lambda_1(\bar{x}_1)^T x_2$$

$$\Rightarrow \lambda_2(\bar{x}_1)^T x_2 = \lambda_1(\bar{x}_1)^T x_2$$

$$\Rightarrow (\lambda_2 - \lambda_1)(\bar{x}_1)^T x_2 = 0$$

$$\Rightarrow (\bar{x}_1)^T x_2 = 0 \quad (\because \lambda_2 \neq \lambda_1)$$

$$\Rightarrow (x_1)^T x_2 = 0$$

Therefore, x_1 and x_2 are orthogonal.

For example, let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. The eigen values of A are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$.

For, $\lambda_1 = 0$, the eigen vector $x_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$, " " " $x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_3 = 2$, " " " $x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Now, $X_1^T X_2 = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot 1 + (-1) \cdot 0 + 1 \cdot 0$

$= 0$

Also, $X_2^T X_3 = 0$, $X_3^T X_2 = 0$.

∴ Therefore, the eigen vectors X_1, X_2 and X_3 are orthogonal.

Theorem: The eigen values of a real orthogonal matrix has a unit modulas.

Proof: Let A be an $n \times n$ real orthogonal matrix and λ be an eigen value of A.

Then $|A - \lambda I_n| = 0$.

Therefore, the homogeneous system of equation $(A - \lambda I_n)X = 0$ has a non-trivial solution.

Say $X = X_1$. Then $AX_1 = \lambda X_1$.

Taking transpose and conjugate, we get –

$$(\overline{AX_1})^T = (\overline{\lambda X_1})^T$$

$$\Rightarrow (\overline{X_1})^T (\overline{A})^T = \bar{\lambda} (\overline{X_1})^T$$

$$\Rightarrow (\overline{X_1})^T A^T = \bar{\lambda} (\overline{X_1})^T, \text{ since } (\overline{A})^T = A^T.$$

Post multiplying by AX_1, we get –

$$(\overline{X_1})^T A^T A X_1 = \bar{\lambda} (\overline{X_1})^T A X_1$$

$$\Rightarrow (\overline{X_1})^T I_n X_1 = \bar{\lambda} (\overline{X_1})^T \lambda X_1$$

$$\Rightarrow (\overline{X_1})^T X_1 = \lambda \bar{\lambda} (\overline{X_1})^T X_1$$

$$\Rightarrow (1 - \lambda\bar{\lambda})(\overline{X_1})^T X_1 = 0$$

$$\Rightarrow 1 - \lambda\bar{\lambda} = 0, \text{ since } (\overline{X_1})^T X_1 \neq 0$$

$$\Rightarrow \lambda\bar{\lambda} = 1$$

$$\Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1.$$

Therefore the eigen values of a real orthogonal matrix has a unit modulas.

Theorem: If every non-zero vector in $\mathbb{R}^n$ is an eigen vector of a real $n \times n$ square matrix A, then A is a scalar matrix.

proof: First- We show that, A has only one repeated eigen values.

Let- x_1 and x_2 be two eigen vectors corresponding to two different eigen values λ_1 and λ_2 of the matrix A.

Then $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.

Since every non-zero vector in $\mathbb{R}^n$ is an eigen vector of the matrix A, $(x_1 + x_2)$ is also the eigen vector of the matrix A.

Let λ_3 be the corresponding eigen value of the eigen vector $(x_1 + x_2)$.

Now, $A(x_1 + x_2) = \lambda_3 (x_1 + x_2)$

$\Rightarrow Ax_1 + Ax_2 = \lambda_3 x_1 + \lambda_3 x_2$

$\Rightarrow \lambda_1 x_1 + \lambda_2 x_2 = \lambda_3 x_1 + \lambda_3 x_2$

$\Rightarrow (\lambda_1 - \lambda_3) x_1 + (\lambda_2 - \lambda_3) x_2 = 0$

$\Rightarrow \lambda_1 - \lambda_3 = 0$ & $\lambda_2 - \lambda_3 = 0$, since x_1 and x_2 are L-I.

$\Rightarrow$

$\therefore \lambda_1 = \lambda_2 = \lambda_3$.

Therefore, the matrix A has only one repeated eigen values, say λ.

Since the rank of the ch. subspace is n, the dim of the eigen values λ is n.

Therefore, by rank-nullity theorem,

$$\dim \ker (A - \lambda I_n) + \dim \operatorname{Im} (A - \lambda I_n) = n$$

$\Rightarrow \dim E_\lambda + \rho(A-\lambda I_n) = n$,

Where $E_\lambda = \{X \in \mathbb{R}^n : (A-\lambda I_n)X = 0$
$= \ker(A-\lambda I_n)\}$

$\Rightarrow$ rank of the ch. subspace + rank of $(A-\lambda I_n) = n$

$\Rightarrow n + \rho(A-\lambda I_n) = n$

$\Rightarrow \rho(A-\lambda I_n) = 0$

$\Rightarrow A = \lambda I_n$

∴ Therefore, A is a scalar matrix.

Theorem: Let A be an $n \times n$ matrix with characteristic polynomial $f(x)$ and β be the eigen value of A. Then $f(\beta)$ is an eigen value of $f(A)$.

Proof: Since β is an eigen value of A, $AX = \beta X$.

Now, $A^2X = \beta(AX) = \beta \cdot \beta X$

$\Rightarrow A^2X = \beta^2 X$ and $A^3X = \beta^2(AX) = \beta^2 \beta X$

$\Rightarrow A^3X = \beta^3 X$

Continuing this process, we get —

$A^KX = \beta^K X$ for $K > 0$.

and so, $f(A)X = f(\beta)X$.

Adding all, $A^{k_i}X = \beta^{k_i}X$, $i_k = 1, 2, \to K$.

we get —

$(A + A^2 + \cdots + A^K)X = (\beta + \beta^2 + \cdots + \beta^K)X$

Therefore $f(\beta)$ is an eigen value of $f(A)$.

$\Rightarrow f(A)X = f(\beta)X$, if $f(x) = x + x^2 + \cdots + x^K$.

Note:

If $x_1, x_2, \to x_K$ be eigen vectors of a $n \times n$ square real matrix A corresponding to the K - distinct eigen values $\lambda_1, \lambda_2 \to \lambda_K$ respectively., then $x_1, x_2, \to x_K$ are linearly independent.

— ○ —

'In the name of Allah'

Lec-4

Defn: An eigen value λ of the matrix A is called the regular eigen value if the algebraic multiplicity of λ and geometric multiplicity of λ are equal.

An eigen value λ is said to be simple if the algebraic multiplicity of λ w.r.t A is equal to one.

Monic polynomial: A polynomial in x over a field F is called a monic polynomial if the co-efficient of the highest power of x is equal to one.

Minimal polynomial: The monic polynomial of the lowest degree which annihilates A is called the minimal polynomial of the matrix A.

Q) Find the minimal polynomial of A, where

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

→ Since the matrix is diagonal, the eigen value of the matrix A are 3, 3, 3.

Therefore, the ch. polynomial is $(x-3)^3$.

Now, the minimal polynomial of A are $(x-3)$, $(x-3)^2$ or $(x-3)^3$.

$$(A-3I_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ Also, $(A-3I_3)^2 = 0$ and $(A-3I_3)^3 = 0$

Since $(x-3)$ is the lowest degree monic polynomial which annihilates A, and therefore, the minimal polynomial of A is $m_A(x) = (x-3)$.

Theorem: The minimal polynomial of a matrix A is unique.

Proof: Suppose, the degree of the minimal polynomial is n. Therefore, there is no non-zero polynomial of the matrix A of degree less than n which annihilates A.

Let $f(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n$

and $g(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$

be two minimal polynomial of the matrix A of degree n.

Since $f(x)$ and $g(x)$ are the minimal polynomial of A, $f(A) = 0$ and $g(A) = 0$

$\therefore f(A) = A^n + b_1 A^{n-1} + b_2 A^{n-2} + \cdots + b_n = 0$ ——— (1)

and $g(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n = 0$ ——— (2)

Subtracting (1) from (2), We get –

$(a_1 - b_1) A^{n-1} + (a_2 - b_2) A^{n-2} + \cdots + (a_n - b_n) = 0$ ——— (3)

From (3) it follows that, the polynomial $(a_1 - b_1) x^{n-1} + (a_2 - b_2) x^{n-2} + \cdots + (a_n - b_n)$ is also annihilates A.

Since its degree less than n it must be a zero polynomial, therefore,

$a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \ldots, a_n - b_n = 0$

i.e, $a_1 = b_1, \quad a_2 = b_2, \ldots, a_n = b_n$.

Thus $f(x) = g(x)$. and this completes the proof.

Theorem: The minimal polynomial of a matrix A is a divisors of every polynomial that annihilates A.

Proof: Suppose, $m_A(x)$ is a minimal polynomial of the matrix A and let $h(x)$ be any polynomial which annihilates A i.e $h(A) = 0$.

Since $h(x)$ and $m_A(x)$ are two polynomial, therefore by Divison algorithem, $\exists$ two polynomial $q(x)$ and $r(x)$ s.t $h(x) = m_A(x)q(x) + r(x)$, ——— (1)

Where the degree of $m_A(x)$ is either zero or less than the degree of the minimal polynomial $m_A(x)$ of the matrix A.

put $x = A$ in (1), We get —

$$h(A) = m_A(A)q(A) + r(A)$$
$$\Rightarrow 0 = 0 + r(A)$$
$$\Rightarrow r(A) = 0.$$

Therefore, the polynomial $r(x)$ annihilates A. If $r(x) \neq 0$, then its degree is smaller than the degree of the minimal polynomial $m_A(x)$ of the matrix A and thus we arrived at a contradiction, that $m_A(x)$ is the minimal polynomial of the matrix. Therefore, $r(x)$ must be zero polynomial.

$\therefore$ From (1), $h(x) = m_A(x)q(x)$

$\Rightarrow m_A(x) \mid h(x)$. □.

Corll: The minimal polynomial of A divides the characteristic polynomial of A.

Q) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$$ (upper triangular matrix)

→ Since the matrix A is upper triangular, the eigen values of the matrix A are 2, 3, 2.

∴ the ch. polynomial of the matrix is

$$\varphi_A(x) = (x-2)^2(x-3).$$

Now, the minimal polynomial of the matrix A is either $(x-2)(x-3)$ or $(x-2)^2(x-3)$.

Now, $(A-2I)(A-3I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$

Also, $(A-2I)^2(A-3I) = 0$

∴ $(x-2)(x-3)$ is the lowest degree monic polynomial which annihilates A.

∴ the minimal polynomial of A is

$$m_A(x) = (x-2)(x-3).$$

Theorem: Let A be an $n \times n$ matrix over a field F, then the characteristic equation and the minimal equation have the same roots & except for multiplicity.

→ Proof: Suppose $\varphi_A(x)$ and $m_A(x)$ denotes the characteristic polynomial of A and the minimal polynomial of A respectively.

We know that, minimal polynomial is a divisor of the ch. polynomial. Then ∃ a polynomial $q(x)$

s.t $\psi_A(x) = m_A(x)\, q(x)$ ——— (1)

First we shall show that, every roots of the minimal polynomial is also a roots of the ch. polynomial.

Let λ is a root of the minimal polynomial of the matrix A. Then $m_A(\lambda) = 0$.

Now, from (1),

$$\psi_A(\lambda) = m_A(\lambda) \cdot q(\lambda)$$
$$\Rightarrow \psi_A(\lambda) = 0$$

$\therefore \lambda$ is a root of the characteristic polynomial $\psi_A(x)$ of the matrix A.

Conversely let, λ is a root of the ch. polynomial $\psi_A(x)$ of A. Then $\exists$ a non-null vector X s.t

$$AX = \lambda X.$$

Since $m_A(x)$ is a polynomial, so

$$m_A(A)X = m_A(\lambda)X, \text{ known theorem.}$$

Since $m_A(x)$ is the minimal polynomial of A,

$$m_A(A) = 0.$$

$$\therefore 0 \cdot X = m_A(\lambda) \cdot X$$
$$\Rightarrow m_A(\lambda) X = 0 \Rightarrow m_A(\lambda) = 0, \text{ since } X \neq 0$$

Therefore λ is a roots of the minimal polynomial $m_A(x)$ of the matrix A. □.

Theorem: Similar matrix have the same minimal polynomial.

Proof: Suppose, A and B be two similar matrix.

Then, $\exists$ a non singular matrix P s.t $B = P^{-1}AP$.

Now, $B^2 = (P^{-1}AP)^2 = (P^{-1}A^2P)(P^{-1}AP)$

$= P^{-1}A I_n A P$

$= P^{-1}A A P = P^{-1}A^2P.$

In a similar way, we can show that,

$B^K = P^{-1}A^KP.$

for every +ve integer K.

Let $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$ be a polynomial of degree n.

Now, $f(A) = A^n + b_1A^{n-1} + b_2A^{n-2} + \cdots + b_nI_n.$

and $f(P^{-1}AP) = f(B)$

$= (P^{-1}AP)^n + b_1(P^{-1}AP)^{n-1} + b_2(P^{-1}AP)^{n-2}$

$+ \cdots + P^{-1}(b_nI_n)P$

since $b_nI_n = P^{-1}(b_nI_n)P.$

$= P^{-1}A^nP + b_1(P^{-1}A^{n-1}P) + b_2(P^{-1}A^{n-2}P)$

$+ \cdots + P^{-1}(b_nI_n)P$

$= P^{-1}(A^n + b_1A^{n-1} + b_2A^{n-2} + \cdots + b_n)AP$

$= P^{-1}f(A)P$

$\Rightarrow f(B) = f(P^{-1}AP)$

$\therefore f(B) = f(P^{-1}AP) = 0$ iff $f(A) = 0.$

Therefore if $f(x)$ is the polynomial of lowest degree which annihilates B then that $f(x)$ is the polynomial of the lowest degree which annihilates A.

$\therefore$ the similar matrix A and B have the same minimal polynomial. □

—— o ——

Matrix Representation of a Linear transformation

Basis: Let $V(F)$ be any vector space, then a basis for V is a linearly independent set of vectors in V which span is equal to the space V. i.e let $A \subseteq V(F)$ then A if A is a basis of V then

(a) A is linearly independent

(b) $\langle A \rangle = L(A) = V(F)$.

Dimention of a vector space: Dimention of a vector space $V(F)$ is the number of vectors in an basis set of the vector space.

For example, the dimention of $R^3(R)$ is 3.

Order basis: Let V be a finite dimentional vector space over the field and order basis of V is a finite squence of L.I vectors in V that generates V

Matrix representation of a L.T:

Let V and W be two vector space over the same field F and let $T: V \to W$ be a linear transformation (or linear mapping) and let $B = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $B' = (\beta_1, \beta_2, \ldots, \beta_m)$ be two basis of the vector space V and W respectively

Let $T(\alpha_1) = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1m}\beta_m$

$T(\alpha_2) = a_{21}\beta_1 + a_{22}\beta_2 + \cdots + a_{2m}\beta_m$

$\vdots$

$T(\alpha_n) = a_{n1}\beta_m + a_{n2}\beta_2 + \cdots + a_{nm}\beta_m$.

The matrix representation of the LT is denoted by $m(T)$ or $[T]_B^{B'}$ and is defined by

$$m(T) = [T]_B^{B'} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

it is also called the matrix of the linear transformation relative to the chosen order basis V and W respectively.

(9) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (4a_1 + a_3,\ 2a_1 + 3a_2 + 2a_3,\ a_1 + 4a_3)$. Find the matrix of T w.r.t standard order basis for $\mathbb{R}^3$ and find the minimal polynomial of the matrix of the linear transformation.

→ Since $\{(1,0,0), (0,1,0), (0,0,1)\}$ is the standard basis of $\mathbb{R}^3$. We have —

$$T(\alpha_1) = T(1,0,0) = (4,2,1)$$
$$= 4(1,0,0) + 2(0,1,0) + 1(0,0,1)$$

$$T(\alpha_2) = T(0,1,0) = (0,3,0)$$
$$= 0(1,0,0) + 3(0,1,0) + 0\cdot(0,0,1)$$

and $T(\alpha_3) = T(0,0,1) = (1,2,4)$
$$= 1(1,0,0) + 2(0,1,0) + 4(0,0,1).$$

Therefore, the matrix of the L.T T is

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}.$$

The ch. equation of A is —

$$|A - xI_3| = \begin{vmatrix} 4-x & 0 & 1 \\ 2 & 3-x & 2 \\ 1 & 0 & 4-x \end{vmatrix} = 0$$

$\Rightarrow \quad (x-3)^2(x-5) = 0$

$\therefore \psi_A(x) = (x-3)^2(x-5)$.

Now, the minimal polynomial of the matrix A is either $(x-3)(x-5)$ or $(x-3)^2(x-5)$

$\because (A-3I)(A-5I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$(A-3I)^2(A-5I) = 0$.

$\therefore (x-3)(x-5)$ is the lowest degree monic polynomial which annihilates A. Therefore, the minimal polynomial of the matrix A is $(x-3)(x-5)$.

(v) Let T be the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = f(1) + f'(0)\cdot x + \{f'(0) + f''(0)\}x^2$. Find the matrix of $P_2(\mathbb{R})$ w.r.t the standard ordered basis β on $P_2(\mathbb{R})$ and find the minimal polynomial of the matrix.

→ Since $\{1, x, x^2\}$ is the standard basis of $P_2(\mathbb{R})$.

We have, $T(1) = 1 + 0 + 0 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = 1 + 1\cdot x + (1+0)\cdot x^2$

$= 1\cdot 1 + 1\cdot x + 1\cdot x^2$

$T(x^2) = 1 + 0 + (0+2)x^2 = 1\cdot 1 + 0\cdot x + 2\cdot x^2$.

Therefore, the matrix representation w.r.t the basis of $P_2(\mathbb{R})$ is $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

$\therefore$ The ch. polynomial of A is

$(A - xI_3) = \begin{vmatrix} 1-x & 1 & 1 \\ 0 & 1-x & 0 \\ 0 & 1 & 2-x \end{vmatrix} = (1-x)^2(2-x)$.

Now, the minimal polynomial of the matrix A is either $(1-x)(2-x)$ or $(1-x)^2(2-x)$.

Now, $(A-I)(A-2I) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$

and also $(A-I)^2(A-2I) = 0$ by Cayley Hamilton theorem.

∴ Since $(x-1)(x-2)$ is the lowest degree monic polynomial which annihilates A, therefore the minimal polynomial of the matrix A is $(x-1)(x-2)$.

Note: Let $T: R^n \to R^m$ be a L.T. Therefore, ∃ an $n \times m$ matrix A s.t $Tx = Ax$.

For example, let $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

$\therefore Tx = Ax$

$\Rightarrow T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \\ x_1 + 4x_3 \end{pmatrix}$

$\Rightarrow T(x_1, x_2, x_3) = (4x_1 + x_3, 2x_1 + 3x_2 + 2x_3, x_2 + 4x_3)$

v) Let T be a linear operator on $\mathbb{R}^3$ defined by $T(x, y, z) = (2x - y, x + y + z, 2z)$. Find the matrix of T w.r.t standard ordered basis β on R^3 and find the minimal polynomial of the matrix.

→ The matrix representation of the linear transformation w.r.t standarded order basis on R^3 is

$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. The ch. polynomial of A is

$|A - xI_3| = \begin{vmatrix} 2-x & -1 & 0 \\ 1 & 1-x & 1 \\ 0 & 0 & 2-x \end{vmatrix}$

$= (2-x)\{x^2 - 3x + 3\}.$

Now, $(2I - A)(A^2 - 3A + 3I) = 0$ by cayley-Hamiltone th.

Since $(2-x)(x^2-3x+3)$ is the lowest degree monic polynomial which annihilates A., therefore the minimal polynomial of the matrix A is

$$m_A(x) = (2-x)(x^2-3x+3).$$

H.W

(i) Find the minimal polynomial of the matrix

(i) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and (ii) $A = \begin{pmatrix} 2 & 1 & 5 & 7 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

(Q) Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f''(x)$. Find the ch. polynomial and the minimal polynomial of the matrix of the linear transformation w.r.t the standard ordered basis β on $P_2(R)$

→ Given, $T(f(x)) = f(x) + (x+1)f''(x)$.

$\therefore T(1) = 1 + 0 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = x + 0 = 0\cdot 1 + 1\cdot x + 0\cdot x^2$

$T(x^2) = x^2 + (x+1)\cdot 2 = 2\cdot 1 + 2\cdot x + 1\cdot x^2$

$\therefore A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. $\therefore \psi_A(x) = (x-1)^3$.

Now, the minimal polynomial of the matrix A is $(x-1)(x-1)^2$ or $(x-1)^3$. Then $(x-I) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$

Now, ~~the minimal polynomial of the matrix A is~~

~~$(x-1)(x-1)^2$~~

But, $(x-I)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Also, $(A-I)^3 = 0$, by C-H theorem.

$\therefore (x-1)^3$ is the lowest degree monic polynomial which annihilates A.

$\therefore$ The minimal polynomial of A is $m_A(x) = (x-1)^3$.

(9) Find the eigen values and the eigen vector of the matrix A, where $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$.

→ The ch. eqn is

$$|A - xI_3| = \begin{vmatrix} 1-x & -3 & 3 \\ 0 & -5-x & 6 \\ 0 & -3 & 4-x \end{vmatrix} = 0$$

$$\Rightarrow (x-1)^2(x+2) = 0$$

$$\Rightarrow x = 1, 1, -2.$$

$\therefore$ The eigen values of the matrix A are 1, 1, −2.

let $\lambda_1 = 1$, $\lambda_2 = -2$.

The A.M of $\lambda_1 = 1$ is 2 and the A.M of $\lambda_2 = -2$ is 1.

and G.M of $\lambda_2 = -2$ is 1.

$\therefore \lambda = -2$ is called the simple eigen value of A because the A.M of $\lambda = -2$ is 1 and also 0,

$\lambda = -2$ is called the regular eigen value of A because A.M = G.M = 1.

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector of the matrix A corresponding to the eigen value $\lambda_1 = 1$ i.e

$$(A - I_3)X = 0$$

$$\Rightarrow \begin{pmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -3x_2 + 3x_3 &= 0 \\ -6x_2 + 6x_3 &= 0 \\ -3x_2 + 3x_3 &= 0 \end{aligned}\right\} \quad \text{---(1)}$$

The system of equations (1) are equivalent to $-x_2 + x_3 = 0$.

i.e $x_2 = x_3 = K$ (say), Where K is an arbitrary non-zero real number.

and $x_1 = c$, c being arbitrary constant.

$$\therefore X = \begin{pmatrix} c \\ K \\ K \end{pmatrix} = cC\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } +K\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Since The eigen vectors are L.I., so geometric multiplicity (G.M) of the eigen value $\lambda = 1$ is 2.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector of A corresponding to the eigen value $\lambda = -2$, $(A + 2I_3)X = 0$

$$\Rightarrow \begin{pmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow 3x_1 - 3x_2 + 3x_3 = 0 \text{ or } x_1 - x_2 + x_3 = 0$$

$$-3x_2 + 6x_3 = 0$$

and $-3x_2 + 6x_3 = 0$

Let $x_3 = K$. $\therefore x_1 = K$, $x_2 = 2K$.

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} K \\ 2K \\ K \end{pmatrix} = K\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = -2$.

Diagonalization of a matrix

A square matrix A is said to be diagonalizable if $\exists$ a non-singular matrix P s.t $P^{-1}AP = D = diag(d_1, d_2, \cdots, d_n)$.

Theorem: An square matrix A is diagonalizable iff A has n L.I eigen vectors.

proof: First we suppose that A is diagonalizable.

Then, $\exists$ a non-singular matrix P s.t

$$P^{-1}AP = D = diag(d_1, d_2, \cdots, d_n).$$

Then, $AP = PD$.

$\Rightarrow Ax_i^* = d_i x_i^*$, where x_i^* denotes the i-th col^m of the matrix P.

Since P is non-singular, so its col^m vectors are L.I.

The col^m vectors of P are the eigen vectors of the matrix A. Therefore, the eigen vectors are of the matrix A are L.I.

Conversely let, $x_1, x_2, \cdots, x_n$ be n L.I eigen vectors of the matrix A corresponding to the eigen values $d_1, d_2, \cdots, d_n$, not necessarily different.

Let P be a square matrix having $x_1, x_2, \cdots, x_n$ as its col^m vectors. Then $Ax_i = d_i x_i$, where $i = 1, 2, \cdots, n$.

$$\therefore AP = PD \quad \text{---------- (1)}$$

Since the col^m vectors of P are L.I, P is non-singular, Therefore P^{-1} exists.

Pre-multiplying (1) by P^{-1}, we get —

$$P^{-1}AP = P^{-1}PD$$
$$= I_n D$$
$$\Rightarrow P^{-1}AP = D.$$

$\therefore$ A is similar to a diagonal matrix D and hence A is diagonalizable by definition.

Theorem: Let A be an $n \times n$ matrix over a field F. If all the eigen values of A be different and belonging to F, then A is diagonalizable but not conversely.

proof: Let $x_1, x_2, \dots, x_n$ be eigen vectors of the matrix A corresponding to the different eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$.

Therefore, the eigen vectors $x_1, x_2, \dots, x_n$ are linearly independent. (known theorem)

Since $x_1, x_2, \dots, x_n$ are n linearly independent eigen vectors of A, therefore, A is diagonalizable.

☑ The converse of the above theorem is not always true.

For example, let $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$.

The eigen values of A are 3, 3, 5.

The ch. polynomial of A is $\varphi_A(x) = (x-3)^2(x-5)$ and the minimal polynomial of A is $m_A(x) = (x-3)(x-5)$.

Since the minimal polynomial is the product of the different linear factors, therefore, A is diagonalizable.

(9) show that, the matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$ is diagonalizable.

→ The ch polynomial of A is

$$|A - xI_3| = (x-1)^2(x+2).$$

Therefore, the eigen values of the matrix A are $1, 1, -2$. Let $\lambda_1 = 1$, and $\lambda_2 = -2$.

∴ the algebraic multiplicity of $\lambda_1 = 1$ is 2 and the geometric multiplicity is 2, because two LI eigen vectors corresponding to the eigen value $\lambda_1 = 1$.

The A.M of $\lambda_2 = -2$ is 1 and G.M = 1,

∴ $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are three L.I eigen vectors of the matrix A.

∴ A is diagonalizable.

Some properties

(1) A matrix A is diagonalizable iff sum of the GM is equal to the order of the matrix A.

(2) A matrix A is diagonalizable iff for each eigen values A.M = G.M

(3) a matrix A is diagonalizable iff its minimal polynomial is a product of different linears factors of the matrix

(4) If all the entries of A are equal then A is diagonalizable.

→ proof let $A = \begin{pmatrix} c & c & \cdots & c \\ c & c & \cdots & c \\ \vdots & & & \\ c & c & \cdots & c \end{pmatrix}$;

0 and ne are the eigen values of A.

The AM of $\lambda = ne$ is $(n-1) = GM$.

The ch. polynomial of A is $x^{n-1}(x - tr(A))$

$= x^{n-1}(x - ne)$.

∴ The minimal polynomial of A is

$x(x - tr(A))$

$= x(x - ne)$.

⑤ Every idempotent matrix can be diagonalizable.

→ Since A is idempotent, $A^2 = A$.

By cayley-Hamilton theorem, the eigen value is $x = 0, 1$. and $n(A) = tr(A)$.

and the ch. polynomial is $x^{n-tr(A)}(x-1)^{tr(A)}$.

⑥ Every involutary matrix can be diagonalizable.

→ Since A is involutary, matrix, $A^2 = I$.

∴ The eigen values of A are $1, -1$.

∴ The ch. polynomial of A is

$(x-1)^{\frac{n+tr(A)}{2}} \cdot (x+1)^{\frac{n-tr(A)}{2}}$.

⑦ Every real symmetric matrix is diagonalizable.

⑧ Every Hermetian matrix is diagonalizable.

⑨ A nilpotent matrix is diagonalizable iff it is a null matrix.

⑩ A square matrix A is diagonalizable iff its all eigen values are regulars.

9) Show that, the matrix $A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$ is diagonalizable.

→ Since the matrix A is upper triangular, the eigen values of the matrix A are 2, 2 & 3.

Therefore, the ch. polynomial of A is

$$\psi_A(x) = (x-2)^2(x-3).$$

and the minimal polynomial of A is

$$m_A(x) = (x-2)(x-3).$$

since the minimal polynomial $m_A(x)$ is the product, ~~therefore~~ different linear factors, therefore, A is diagonalizable.

Alternative Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vectors of A corresponding to the eigen value $\lambda = 2$.

$$(A - 2I_3)X = 0$$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 14 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -2x_2 + 14x_3 &= 0 \\ x_2 - 7x_3 &= 0 \end{aligned}\right\} \text{Which is equivalent to } x_2 = 7x_3.$$

Let $x_3 = K$, then $x_2 = 7K$; K being a non-zero real number and let $x_1 = c$, c is arbitrary non-zero real no.

$$\therefore X = \begin{pmatrix} c \\ 7K \\ K \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + K\begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$ are two L.I eigen vectors of A corresponding to $\lambda = 2$.

$\therefore$ G.M of $\lambda = 2$ is 2.

$\therefore$ G.A.M = G.M for the eigen value $\lambda = 2$.

Also, A.M = G.M = 1, for the eigen value $\lambda = 3$.

Hence, A is diagonalizable (by one property).

Alternative

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\7\\1\end{pmatrix}$ are two L.I eigen vectors of A corresponding to the eigen value $\lambda = 2$.

Let $X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ be the eigen vector of A corresponding to the eigen value $\lambda = 3$ i.e

$$(A - 3I_3)X = 0$$

$$\Rightarrow \begin{pmatrix}-1 & -2 & 14\\ 0 & 0 & -7\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0$$

$$\Rightarrow -x_1 - 2x_2 + 14x_3 = 0 \quad \text{①}$$

$$-7x_3 = 0 \Rightarrow x_3 = 0$$

$$-x_3 = 0 \Rightarrow x_3 = 0$$

From ①, $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

Let $x_2 = K$. Then $x_1 = -2K$.

$$\therefore X = \begin{pmatrix}-2K\\K\\0\end{pmatrix} = K\begin{pmatrix}-2\\1\\0\end{pmatrix}.$$

$\therefore \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\7\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\0\end{pmatrix}$ are the three L.I eigen vectors of A, therefore, the matrix A is diagonalizable.

(v) Show that, the matrix $A = \begin{pmatrix}4 & -1\\12 & -3\end{pmatrix}$ is diagonalizable

→ The ch. equation of A is

$$|A - xI_2| = 0$$

$$\Rightarrow (x-4)(x+3) + 12 = 0$$

$$\Rightarrow x^2 = x.$$

$\therefore$ By cayley-Hamilton theorem, $A^2 = A$.

Since the matrix A is idempotent, the matrix A is diagonalizable.

d) show that, the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ is diagonalizable.

→ Since the matrix A is lower triangular, the eigen values of A are $1, 1, -1$.

The ch. equation of A is $\psi_A(x) = (x-1)^2(x+1)$ and the minimal polynomial of A is either $(x-1)(x+1)$ or $(x-1)^2(x+1)$.

Now, $(A-I)(A+I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$.

Also, $(A-I)^2(A+1) = 0$, by cayley-Hamilton theorem.

Since $(x-1)(x+1)$ is the lowest degree monic polynomial which annihilates A, therefore $(x-1)(x+1)$ is the minimal polynomial.

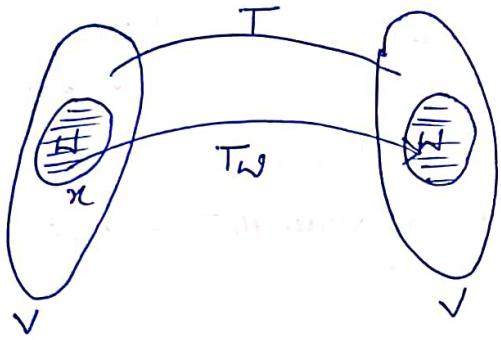
Since the minimal polynomial is the product of different linear factors, therefore A is diagonalizable.

Note Here, $A^2 = A \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$.

Since A is involutory matrix, therefore A is diagonalizable.

e) H.W Show that, the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ is not diagonalizable.

Invariant subspace: Let V be a vector space over the field F and $T: V \to V$ be a linear operator. A subspace W of V is said to be invariant under T if $T(W)$ in W i.e $T(x) \in W \quad \forall x \in W$.



Note: If W is a invariant subspace of V then the restriction operator T_W is a linear operator on W i.e if $T: V \to V$ and $T(W) \subseteq W$, then $T_W : W \to W$ is a linear operator.

Theorem: The range space and null space of a linear transformation $T: V \to V$ is invariant subspace under T.

Proof: $\text{Ker}\, T = N(T) = \{\alpha \in V : T(\alpha) = 0'\}$.

Let $\alpha \in \text{Ker}\, T$. Then $T(\alpha) = 0'$, here $0'$ being the null element in V.

We will show that, $\alpha \in \text{Ker}\, T \Rightarrow T(\alpha) \in \text{Ker}\, T$.

Now, $T(T(\alpha)) = T(0') = 0'$.

Therefore, $T(\alpha) \in \text{Ker}\, T$. So, $\alpha \in \text{Ker}\, T \Rightarrow T(\alpha) \in \text{Ker}\, T$.

$\therefore$ $\text{Ker}\, T$ is an invariant subspace of V.

We know that, Range of $T = \{w : T(v) = w, \; v \in V\}$.

$w \in R(T) \Rightarrow T(v) = w$, $\forall v \in V$ by definition.

$\therefore T(w) \in R(T)$, so $w \in R(T) \Rightarrow T(w) \in R(T)$

$\therefore R(T)$ is a invariant subspace of V.

Ex: Let T be a linear operator on a vector space V. Then the following subspace of V are invariant subspace.

(i) $\{0\}$. since $T(0) = 0 \subseteq \{0\}$ (trivial invariant subspace).

(ii) Since $T: V \to V$, $T(v) \subseteq V$, therefore V is an improper invariant subspace.

(iii) Eigen space,

$$E_\lambda = \{ v \in V : Tv = \lambda v \}.$$

proof: Let $x \in E_\lambda$. Then $T(x) = \lambda x \in E_\lambda$;

since $T(\lambda x) = \lambda T(x) = \lambda(\lambda x)$

$\therefore T(E_\lambda) \subseteq E_\lambda$. Hence E_λ is an invariant subspace of V.

(iv) Theorem: Let W_1 and W_2 be two invariant subspace of a vector space under T. Then $W_1 \cap W_2$ is an T-invariant subspace of V.

proof: Let $W = W_1 \cap W_2$. Then $x \in W \Rightarrow x \in W_1 \cap W_2$

$\Rightarrow x \in W_1$ and $x \in W_2$

$\Rightarrow T(x) \in W_1$ and $T(x) \in W_2$

$\Rightarrow T(x) \in W_1 \cap W_2$

$\Rightarrow T(x) \in W$.

Thus $x \in W \Rightarrow T(x) \in W$.

$\therefore W_1 \cap W_2$ is an T-invariant subspace.

Def^n: Let V be a vector space and $T: V \to V$ be a linear operator and x be a non-zero element in V, then

$$W = \text{span}\{x, T(x), T^2(x), T^3(x), \ldots\}$$ is called the T-cyclic subspace of V generated by x.

Theorem: Let V be a vector space over a field and T be a linear operator of V. Then T-cyclic subspace generated by a non-zero vector v is the smallest T-invariant subspace of V containing v.

proof: Let W be the T-cyclic subspace of V generated by v. Then

$$W = \text{span}\{v, T(v), T^2(v), T^3(v), \ldots\}$$

To show, (i) W is a subspace of V containing v.

(ii) W is an T-invariant subspace of V.

(iii) W is the smallest T-invariant subspace of V containing v.

clearly W is a subspace of V.

Let $w \in W$. Then $w = c_0 v + c_1 T(v) + c_2 T^2(v) + \cdots$

$$= \sum_{i=0}^{\infty} c_i T^i(v)$$

Now, $T(w) = T\left(c_0 v + c_1 T(v) + c_2 T^2(v) + \cdots\right)$

$$= T(c_0 v) + T(c_1 T(v)) + T(c_2 T^2(v)) + \cdots$$

$$= c_0 T(v) + c_1 T^2(v) + c_2 T^3(v) + \cdots$$

$$= \sum_{i=0}^{\infty} c_i T^{i+1}(v) \in W.$$

Thus, $w \in W \Rightarrow T(w) \in W$.

Therefore, W is an invariant subspace of V.

Let S be a invariant subspace of V containing v.

To show, $W \subseteq S$. Since S is a invariant subspace of V containing v. $\Rightarrow T(v) \in S, \; v \in S$

$\Rightarrow T^i(v) \in S, \; i = 1, 2, \cdots$

$\Rightarrow w \in S$

$T^i(v) \in S$
$\Rightarrow c_i T^i(v) \in S$
$\Rightarrow \sum c_i T^i(v) \in S$
$\Rightarrow w \in S.$

Thus, $w \in W \Rightarrow w \in S$

$\therefore W \subseteq S.$

$\therefore W$ is the smallest T-invariant subspace of V containing v.

Ex: Let T be the linear operator on $\mathbb{R}^3$ defined by $T(a,b,c) = (-b+c, a+c, 3c)$. Find the T-cyclic subspace generated by $e_1 = (1,0,0)$.

Sol^n: We have, $T(e_1) = T(1,0,0) = (0,1,0) = e_2$

$T^2(e_1) = T(T(e_1)) = T(e_2) = T(0,1,0) = (-1,0,0) = -e_1$

Also, $T^3(e_1) = T(T^2(e_1)) = T(-e_1) = (0,-1,0) = -e_2$

It follows that,

$\text{Span} = [e_1, T(e_1), T^2(e_1), T^3(e_1), \cdots]$

$= \{e_1, e_2\} = \{(1,0,0), (0,1,0)\}$

$= \{s(1,0,0) + t(0,1,0) : s,t \in \mathbb{R}\}$

$= \{(s,t,0) : s,t \in \mathbb{R}\}.$

ⓒ Let T be the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = f'(x)$, then the T-cyclic subspace generated by x^2 is

$\text{Span} = \{x^2, 2x, 2\} = P_2(\mathbb{R}).$

Soln We have, $T(x^2) = 2x$,

$$T^2(x^2) = T(T(x^2)) = T(2x) = 2$$

$$T^3(x^2) = T(2) = 0.$$

It follows that,

$$\text{span} = \{x^2, T(x^2), T^2(x^2), T^3(x^2), \ldots\} = \{x^2, 2x, 2\} = P_2(\mathbb{R}).$$

(9) Let T be the linear operator on $\mathbb{R}^4$ defined by $T(a,b,c,d) = (a+b, b+2c-d, b+d, 2c-d, c+d)$ and let $W = \{(t,s,0,0) : t,s \in \mathbb{R}\}$. To show that, $T(W) \subseteq W$.

→ Let $(x,y,0,0) \in W$.

Then $T(x,y,0,0) = (x+y, y, 0, 0) \in W$.

So, $(x,y,0,0) \in W \Rightarrow T(x,y,0,0) \in W$.

Thus, $T(W) \subseteq W$. ⇒ W is a invariant subspace.

Note: Let $\gamma = (e_1, e_2)$ is an order basis for W. extended γ to the standard order basis $\beta = \{e_1, e_2, e_3, e_4\}$ for V.

Let $A = [T]_\beta$ and $B = [T_W]_\gamma$.

Now, $T(1,0,0,0) = (1,0,0,0) = 1(1,0,0,0) + 0(0,1,0,0)$

$T(0,1,0,0) = (1,1,0,0) = 1(1,0,0,0) + 1(0,1,0,0)$

$\therefore B = [T_W]_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Again, $T(1,0,0,0) = (1,0,0,0) = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4$

$T(0,1,0,0) = (1,1,0,0) = 1 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4$.

$$T(0,0,1,0) = (2,0,2,1)$$
$$= 2e_1 + 0\cdot e_2 + 2e_3 + e_4$$
$$T(0,0,0,1) = (-1,1,-1,1)$$
$$= -e_1 + 1\cdot e_2 - 1e_3 + 1\cdot e_4$$

$$\therefore A = [T]_\beta = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_{2\times 2} & \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}_{2\times(4-2)} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{4-2\times 2} & \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}_{4-2\times(4-2)} \end{pmatrix}$$

Let $f(t)$ and $g(t)$ be the polynomial of A and B respectively.

$$|A - tI_4| = \begin{vmatrix} 1-t & 1 & 2 & -1 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 2-t & -1 \\ 0 & 0 & 1 & 1-t \end{vmatrix}$$

$$= (1-t) \begin{vmatrix} 1-t & 0 & 1 \\ 0 & 2-t & -1 \\ 0 & 1 & 1-t \end{vmatrix}$$

$$= (1-t)\left[(2-t)(1-t)+1\right].$$

and $$|B - tI_2| = \begin{vmatrix} 1-t & 1 \\ 0 & 1-t \end{vmatrix} = (1-t)^2.$$

The ch. polynomial of B is $g(t) = (1-t)^2$

$\therefore f(t) = (1-t)^2 \cdot [(2-t)(1-t)+1]$

$\Rightarrow f(t) = g(t) \cdot [(2-t)(1-t)+1]$

$\Rightarrow g(t) \mid f(t).$

Lec-8

Theorem: Let T be a linear operator on a finite dimensional vector space V and W be a T-invariant subspace of V then the characteristic polynomial of T_W divides the characteristic polynomial of T.

proof: choose an order basis $\gamma = (\alpha_1, \alpha_2, \dots, \alpha_n)$ for W. and extended it to an order basis

$$\beta = (\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n) \text{ for } V.$$

Let $A = [T]_\beta$ and $B = [T_W]_\gamma$.

Since W is T-invariant subspace, $T(W) \subseteq W$.

Then, A can be written in the form

$$A = \begin{pmatrix} B_{k\times k} & B_{k\times(n-k)} \\ O_{(n-k)\times k} & C_{(n-k)\times(n-k)} \end{pmatrix}$$

Let $f(t)$ be the characteristic polynomial of T and $g(t)$ be the characteristic polynomial of T_W.

$$\text{Then } |A - tI_n| = \begin{vmatrix} |B - tI_{k\times k}| & B_{k\times(n-k)} \\ O_{(n-k)\times k} & |C - tI_{(n-k)\times(n-k)}| \end{vmatrix}$$

$$= |B - tI_{k\times k}| \, |C - tI_{(n-k)\times(n-k)}|$$

Now, the characteristic polynomial of the matrix B is

$$|B - I_{k\times k}| = g(t).$$

Therefore, $|A - tI_n| = |B - tI_{k\times k}| \, |C - tI_{(n-k)\times(n-k)}|$

$$\Rightarrow f(t) = g(t) \, |C - tI_{(n-k)\times(n-k)}|$$

$$\Rightarrow g(t) \mid f(t).$$

Hence the ch. polynomial of T_W divides the characteristic polynomial of T.

Theorem: The eigen vector of a linear operator $T: V \to V$ may be characterised as generator of T-invariant one-dimentional subspace.

proof: Let u be an eigen vector of the linear operator T corresponding to the eigen value λ.

Therefore, $Tu = \lambda u$

let $W = \text{span}\{u\}$.

Now, $T(au) = aT(u)$, since T is linear operator

$= a(\lambda u)$

$= (a\lambda)u$

$= cu \in W$ where $c = a\lambda$.

$\therefore T(W) \subseteq W$

Therefore, W is a one-dimentional T-invariant subspace.

Conversely, let, W is a one-dimentional T-invariant subspace of V.

We will show that, W is generated by an eigen vector of T.

let $W = \text{span}\{au\}$ and W is T-invariant.

Then $T(au) \in W$

$\Rightarrow T(au) = \beta u$

$\Rightarrow aT(u) = \beta u$

$\Rightarrow T(u) = (a^{-1}\beta)u$

$\Rightarrow T(u) = \lambda u, \quad \lambda = a^{-1}\beta$.

e) let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Find all the T-invariant subspace of $\mathbb{R}^2(\mathbb{R})$.

→ Since the linear operator T is given by the matrix $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$.

Now, $|A - tI_2| = \begin{vmatrix} 2-t & -5 \\ 1 & -2-t \end{vmatrix} = t^2 + 1$.

The ch. equation is $|A - tI_2| = 0$

$\Rightarrow t^2 + 1 = 0 \Rightarrow t = \pm i$

Therefore, no eigen vector of the matrix A.

Hence, No one-dimentional invariant subspace of $R^2(R)$ exists.

Therefore, the 0-dimentional subspace is $\{0\}$ and the two dimentional subspace is R^2.

Q) Let $T: R^2 \longrightarrow C^2$ be a linear operator defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Find all the T-invariant subspace of $C^2(C)$.

→ Since the linear operator T is given by the matrix $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$. Now, $|A - tI_2| = \begin{vmatrix} 2-t & -5 \\ 1 & -2-t \end{vmatrix}$

$= t^2 + 1$.

The ch. equation is $|A - tI_2| = 0$ i.e $t^2 + 1 = 0$

$\Rightarrow t = \pm i$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen ~~value~~ vector of the matrix A corresponding to the eigen value $\lambda_1 = i$.

$\therefore (A - \lambda I_2) X = 0$

$\Rightarrow (A - \lambda i)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow (2-i)x_1 - 5x_2 = 0$$

$$x_1 - (2+i)x_2 = 0$$

$$\Rightarrow x_1 = (2+i)x_2 \quad \text{---------- (1)}$$

let $x_2 = c$, where c is arbitrary non-zero real no.

$\therefore x_1 = (2+i)c$.

$$\therefore X = \begin{pmatrix} (2+i)c \\ c \end{pmatrix} = c \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

let $W = \langle (2+i, 1) \rangle$.

Therefore, W_1 is a 1-dimentional T-invariant subspace.

Similarly, we get $X = k \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$ is an eigen vector of the matrix A corresponding to the eigen value $\lambda_2 = -i$

let $W_2 = \langle 2-i, 1 \rangle$.

Therefore W_2 is a one dimentional T-invariant subspace. Also, 0 dimentional subspace is $\{0\}$ and two dimentional subspace is $\mathbb{C}^2(\mathbb{C})$.

Inner product space:

let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two vectors in $\mathbb{R}^2$.

Therefore, by Pythagoras theorem,

$$(OP)^2 = (OA)^2 + AP^2$$

$$= x_1^2 + x_2^2 \quad \Rightarrow OP = \sqrt{x_1^2 + x_2^2}.$$

We defined the length/norm of x as positive $\sqrt{x_1^2+x_2^2}$ and it is denoted by $\|x\|$.

By pythagoras theorem,

$$PQ^2 = AQ^2 + PA^2$$
$$= (x_1-y_1)^2 + (x_2-y_2)^2$$
$$\Rightarrow PQ = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$
$$= \|x-y\|.$$

Let $d(x-y) = \|x-y\|$.

$d(x,y)$ gives distance between two points.

Definition: The dot product of x and y is denoted by $\langle x \cdot y \rangle$ or $x \cdot y$ and is defined as

$$\langle x \cdot y \rangle = x_1y_1 + x_2y_2.$$

Note:

① If $y = x$ then $\langle x, x \rangle = x_1 \cdot x_1 + x_2 \cdot x_2$

$$\Rightarrow \|x\|^2 = x_1^2 + x_2^2$$
$$\Rightarrow \|x\| = \sqrt{x_1^2 + x_2^2}.$$

Definition: An inner product on a dot product on a vector space V over the field F is a map

$\langle \cdot \rangle : V \times V \to F$ satisfying the following properties. For $x, y, z \in V$ and $\alpha, \beta \in F$

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

(iii) $\langle \alpha x + \beta z, y \rangle = \alpha \langle x, y \rangle + \beta \langle z, y \rangle$.

A vector space V together with an inner product is called an inner product space.

A real inner product space is called a Euclidian space and a complex inner product space is called a unitary inner product space.

Example:

① Let $x = (x_1, x_2 \to x_n)$ and $y = (y_1, y_2 \to y_n)$ be two vectors in $\mathbb{R}^n$. Then inner product

$$\langle x \cdot y \rangle = \sum_{i=1}^{n} x_i y_i \text{ on } \mathbb{R}^n.$$

because

(i) $\langle x \cdot x \rangle = \sum_{i=1}^{n} x_i^2 \geq 0$ and $\langle x \cdot x \rangle = 0$ iff $x = 0$.

(ii) $$\langle x \cdot y \rangle = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \langle y \cdot x \rangle.$$

(iii) $$\langle \alpha x + \beta z, y \rangle$$

$$= \sum_{i=1}^{n} (\alpha x_i + \beta z_i) \cdot y_i$$

$$= \sum_{i=1}^{n} \alpha (x_i y_i) + \sum_{i=1}^{n} \beta (z_i y_i)$$

$$= \alpha \sum_{i=1}^{n} (x_i y_i) + \beta \sum_{i=1}^{n} (z_i y_i) = \alpha (x, y) + \beta (z, y)$$

This inner product is called the standard inner product on $\mathbb{R}^n$.

② on $\mathbb{C}^n$: $\langle x \cdot y \rangle = \sum_{i=1}^{n} \overline{x_i} y_i$, this inner product is called the standard inner product on $\mathbb{C}^n$.

Q) For $x, y \in \mathbb{R}^2$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\langle x \cdot y \rangle = y_1(x_1 + 2x_2) + y_2(2x_1 + 5x_2)$ defines an inner product. We shall prove that, if $\langle x \cdot x \rangle = 0$ then $x = 0$.

→ We have, $\langle x \cdot x \rangle = x_1(x_1 + 2x_2) + x_2(2x_1 + 5x_2)$

$= x_1^2 + 2x_1x_2 + 2x_1x_2 + 5x_2^2$

$= x_1^2 + 4x_1x_2 + (2x_2)^2 + x_2^2$

$= (x_1 + 2x_2)^2 + x_2^2$

Now, $0 = (x_1 + 2x_2)^2 + x_2^2$

$\therefore x_2 = 0$ and $x_1 + 2x_2 = 0 \Rightarrow x_1 = 0$

$\therefore x = (x_1, x_2) = (0, 0) = 0$.

Defⁿ: If $x = (x_1, x_2, \dots, x_n)$ be the vectors of $\mathbb{R}^n$ then the length / norm of the vector x is denoted by $\|x\|$ and is defined as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \left[\sum_{i=1}^{n} (x_i)^2\right]^{\frac{1}{2}}.$$

For $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ define the distance between two points x and y by

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

$$= \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}}.$$

Cauchy-Schawarz's inequality:

For any two vectors α, β in a Euclidian space V, $|(\alpha, \beta)| \leq \|\alpha\| \|\beta\|$. The inequality holds when α and β are Linearly dependent.

proof: Case-1: Let one or both α and β be null then both sides equality holds.

Case-2: Let α and β ~~two~~ be two L.D

vectors.

Then $\exists$ a real number k s.t $\alpha = k\beta$.

Now, $\|\alpha\| = \|k\beta\| = |k| \|\beta\|$. ——— (1)

We have, $(\alpha, \beta) = (k\beta, \beta)$

$= k(\beta, \beta)$

$= k \cdot \|\beta\|^2$

$= k \cdot \|\beta\| \cdot \|\beta\|$

Then, $|(\alpha, \beta)| = |k| \|\beta\| \|\beta\|$

$\Rightarrow |(\alpha, \beta)| = \|\alpha\| \|\beta\|$

Case-III: Let α and β be two non-null not L.D vectors. Then $\alpha - k\beta \neq 0$ $\forall$ real values k.

Then, $\langle \alpha - k\beta, \alpha - k\beta \rangle > 0$

$\Rightarrow \|\alpha\|^2 - k(\alpha, \beta) - k(\beta, \alpha) + k^2(\beta, \beta) > 0$

$\Rightarrow \|\alpha\|^2 - 2k(\alpha, \beta) + k^2 \|\beta\|^2 > 0$

$\Rightarrow k^2 \|\beta\|^2 - 2k(\alpha, \beta) + \|\alpha\|^2 > 0$

This is a quadratic equation in k.

The L.H.S is greater than 0 if the discriminant is less than 0. i.e

$4(\alpha, \beta)^2 - 4\|\beta\|^2 \|\alpha\|^2 < 0$

$\Rightarrow (\alpha, \beta)^2 - \|\beta\|^2 \|\alpha\|^2 < 0$

$\Rightarrow (\alpha, \beta)^2 < \|\beta\|^2 \|\alpha\|^2$

$\Rightarrow |(\alpha, \beta)| < \|\alpha\| \|\beta\|$

This completes the proof.

Lemma:

Let V be an inner product space, the norm ~~matrix~~ function

$\|x\| : V \to \mathbb{R}$ has the following three properties —

(i) $\|x\| > 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, i.e. $x \in V$, $\alpha \in \mathbb{R}$.

(iii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

proof: (i) If $x \neq 0$, $(x;x) > 0 \Rightarrow \|x\| > 0$ and if $x = 0$, $(x;x) = 0 \Leftrightarrow \|x\| = 0$

(ii) $\|\alpha x\|^2 = (\alpha x, \alpha x)$

$= \alpha (x, \alpha x)$

$= \alpha^2 (x;x)$

$= \alpha \|x\|^2$

$\therefore \|\alpha x\| = |\alpha| \, \|x\|$

(iii) $\|\alpha + \beta\|^2 = (\alpha + \beta, \alpha + \beta)$

$= (\alpha;\alpha) + (\alpha;\beta) + (\beta;\alpha) + (\beta;\beta)$

$= \|\alpha\|^2 + 2(\alpha;\beta) + \|\beta\|^2$

$\leq \|\alpha\|^2 + 2\|\alpha\|\|\beta\| + \|\beta\|^2$

[by cauchy - schwantz's inequality]

$= (\|\alpha\| + \|\beta\|)^2$

$\therefore \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$, This inequality is called the triangular inequality.

—o—